JOURNAL OF SPACECRAFT AND ROCKETS
Vol. 33, No. 6, November–December 1996

Engineering Notes

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Stability of a Deformed Finite Length Inviscid Liquid Column

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Introduction

IN addition to being of fundamental interest, the finite length liquid column (liquid bridge) has been utilized as a model for float zones. With a microgravity environment available through the Space Shuttle or rocket flights, opportunities for experimental efforts give impetus to studies of liquid column fluid dynamics. Static stability limits and equilibrium shapes of the finite length column are well known.[1] Studies of the static stability of a modified column have been done, including determination of static stability limits for the finite length column between coaxial, unequal radii end disks with a body force directed along the longitudinal axis of the column[2] and for cases in which the end disks were not coaxial and/or a constant body force was directed perpendicular to the longitudinal axis of the column.[3] These modifications act to reduce the column's static stability limit.

The natural modes of oscillation of the finite length liquid column have been determined. In a linearized, inviscid analysis, natural frequencies were determined for axisymmetric[4] and nonaxisymmetric[5] oscillations. Accompanying experimental work verified the analytical results for a range of slenderness parameter values for the two lowest axial modes[4] and for the helical mode.[5]

In the present microgravity laboratories, the acceleration environment can be above desired levels and can involve both steady and time-dependent components. This situation has prompted studies of fluid column response to periodic forcing directed along the column's longitudinal axis.[6,7] The present work investigates the stability of a deformed finite length liquid column to time-dependent perturbations. The deformed column shape is due to a steady body force oriented normal to the column's longitudinal axis.

Formulation

Governing equations are conservation of mass and momentum. The analysis is inviscid, linear, and incompressible. Time $\tilde{t}$ and the spatial variable $\tilde{x}$ are nondimensionalized by $R_0(R_0\rho/\gamma)^{1/2}t = \tilde{t}$ and $R_0\underline{x} = \tilde{x}$, respectively, with the tilde denoting a dimensional quantity. Surface tension is given by γ and density by ρ. The term R_0 is the radius of the undeformed, undisturbed liquid column. Velocity $\underline{u}$ and pressure p nondimensionalize as $(\gamma/\rho R_0)^{1/2}\underline{u} = \tilde{u}$ and $(\gamma/R_0)p = \tilde{p}$. Nondimensional equations (using cylindrical coordinates) are

$$\nabla \cdot \underline{u} = 0 \tag{1a}$$

$$\frac{\partial u}{\partial t} + u \cdot \nabla(\underline{u}) = -\nabla p + B\nabla[r\cos(\theta)] \tag{1b}$$

The coefficient of the lateral forcing term is $B = \rho G_0 R_0^3/\gamma$, where G_0 is the magnitude of imposed steady-state forcing. The term B serves as a measure of deformation of the static column from a circular cylinder; the larger B is, the more deformed the column. Let the longitudinal axis of the column be aligned with the vertical direction (z coordinate). The column is capped by two end disks, considered fixed in space. The liquid does not spill over around the top of the end disks. The shape of the column is that of a "bowed" right circular cylinder. The larger the lateral forcing, the more the liquid column will assume a bowed, "c" shape. The two end disks that form the boundaries of the column in the vertical direction are located at the nondimensional z value of $\pm\Lambda(= L/2R_0)$, where L is a dimensional column length, and R_0 is the dimensional radius of a slice of the static column in a constant z plane. A typical column radius is 0.5 cm, and a typical length is 2 cm.

Boundary conditions at the end disks are the anchored triple contact line condition and zero normal velocity. At the liquid column surface, the kinematic condition and normal force balance are imposed. Deformations must satisfy conservation of volume.

The magnitude of B is assumed small. A velocity potential is defined. The system is perturbed about a state of zero mean motion with dependent quantities expanded in a series with expansion parameter δ. Use of the velocity potential in Eq. (1a) yields

$$\nabla^2\phi = 0 \tag{2}$$

where ϕ is the velocity potential. This is solved in the cylindrical domain of $0 \le r \le 1$, $0 \le \theta < 2\pi$, and $-\Lambda \le z \le \Lambda$. Modifications due to the deformation appear in the interface conditions. The solution that satisfies zero normal velocity on the end disks is

$$\phi(r,\theta,z,t) = \sum_{m=0}^{\infty} A_{m,0} r^m e^{im\theta} e^{\lambda t} + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} A_{m,n} I_m(l_n r)\cos[l_n(z+\Lambda)]e^{im\theta}e^{\lambda t} \tag{3}$$

with $A_{m,n}$ the unknown coefficients and λ the unknown frequency. The slenderness parameter Λ is $(L/2R_0)$, with L the column length. Larger values of Λ indicate more slender columns. The term $I_m(l_n r)$ is a modified Bessel function of order m; $l_n = (n\pi)/(2\Lambda)$, and n is an integer.

Let the equilibrium free surface Fe be defined by

$$Fe = r - [1 + Bg_{0100}(r,\theta) + \delta\zeta(\theta,z,t)] = 0 \tag{4}$$

where ζ is the time-dependent perturbation to the interface, and

$$\zeta = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} H_{m,n}\sin\left(\frac{n\pi z}{\Lambda}\right)e^{im\theta}e^{\lambda t} + \sum_{m=0}^{\infty} G_m z^2 e^{im\theta}e^{\lambda t} + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} E_{m,n}\cos\left(\frac{n\pi z}{\Lambda}\right)e^{im\theta}e^{\lambda t} \tag{5}$$

The deformation of the free surface due to lateral, steady forcing is given by terms that are multiplied by B. Unknown coefficients are $H_{m,n}$, $G_{m,n}$, and $E_{m,n}$. The time-dependent perturbation in Eq. (4), $\delta\zeta(\theta,z,t)$, generally does not satisfy the anchored triple contact line condition on the end disks, which requires that $\zeta = 0$. A tau approximation will be used in the implementation of the spectral method.[8] Both B and δ are small; it is further assumed that $\delta \ll B$. The function g_{0100} is known and is $\{\cos\theta\,(\pi^2/2)[1-(z/\Lambda)^2]\}$ (Ref. 3).

Received Jan. 10, 1996; revision received July 1, 1996; accepted for publication July 5, 1996.

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A Taylor series expansion about $r = 1$ transfers the surface at which kinematic and normal force balance conditions are imposed to the cylindrical surface at $r = 1$. In this linearized analysis, terms that contribute are of order δ and δB. Without the lateral forcing and resulting deformation of the column from a circular cylindrical shape, terms of order δB would not occur. The linearized kinematic and normal force balance equations are

$$\frac{\partial \zeta}{\partial t} = \frac{\partial \phi}{\partial r} + B\left[\frac{g\partial^2\phi}{\partial r^2} - \left(\frac{\partial g}{\partial z}\right)\left(\frac{\partial \phi}{\partial z}\right) - \left(\frac{\partial g}{\partial \theta}\right)\left(\frac{\partial \phi}{\partial \theta}\right)\right] \tag{6}$$

$$\left(\zeta + \frac{\partial^2\zeta}{\partial\theta^2} + \frac{\partial^2\zeta}{\partial z^2}\right) - B\left[g\zeta + \left(\frac{\partial g}{\partial \theta}\right)\left(\frac{\partial \zeta}{\partial \theta}\right) - \left(\frac{\partial g}{\partial z}\right)\left(\frac{\partial \zeta}{\partial z}\right)\right.$$
$$\left. + 2g\left(\frac{\partial^2 g}{\partial\theta^2}\right)\right] + B(\cos\theta)\zeta = \frac{\partial\phi}{\partial t} + B\frac{g\partial^2\phi}{\partial r\partial t} \tag{7}$$

where the subscript on g_{0100} has been dropped for compactness.

This problem reduces to two subproblems.

1) In the odd subproblem, the time-dependent perturbations are given by

$$\zeta_{\text{odd}}(\theta, z, t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} H_{m,n}\sin\left(\frac{n\pi z}{\Lambda}\right)e^{im\theta}e^{\lambda t} \tag{8}$$

Clearly, the anchored triple contact line condition $\zeta|_{z=\pm\Lambda} = 0$ is satisfied identically for ζ_{odd}. Conservation of volume is

$$\int_0^{2\pi}\int_{-\Lambda}^{\Lambda}\zeta_{\text{odd}}\,\mathrm{d}z\,\mathrm{d}\theta = 0; \qquad B\int_0^{2\pi}\int_{-\Lambda}^{\Lambda} g_{0100}\,\zeta_{\text{odd}}\,\mathrm{d}z\,\mathrm{d}\theta = 0 \tag{9}$$

These conditions are identically satisfied.

Expansions for $\{\zeta_{\text{odd}}, \phi\}$ are substituted into Eqs. (6) and (7) and orthogonality properties of Fourier series utilized. The partial differential equations are transformed into an infinite system of algebraic equations in the unknown coefficients $A_{m,2n-1}$ and $H_{m,n}$ and eigenfrequencies $\lambda_{m,n}$. Orthogonality properties serve to select the antisymmetric (with respect to z) contribution from ϕ. The mth azimuthal modes couple to the $(m \pm 1)$ modes as long as $B \neq 0$. Nonaxisymmetry in θ is intrinsic to the laterally forced ($B \neq 0$) column. Truncations of (NMAX, MMAX) are used; MMAX (NMAX) is the largest azimuthal m mode (number of axial modes) retained. This truncation yields 2(NMAX)(MMAX + 1) algebraic equations, half due to normal force balance, half to the kinematic condition.

2) In the even subproblem, time-dependent perturbations are

$$\zeta_{\text{even}}(\theta, z, t) = \sum_{m=0}^{\infty} G_m z^2 e^{im\theta}e^{\lambda t} + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} E_{m,n}\cos\left(\frac{n\pi z}{\Lambda}\right)e^{im\theta}e^{\lambda t} \tag{10}$$

Development of the eigensystem proceeds similarly. Here, however, the contact line condition at $z = \pm\Lambda$ is not identically satisfied. A tau approximation[8] is used, resulting in

$$G_m = -\left(\frac{1}{\Lambda^2}\right)\sum_{n=0}^{\infty} E_{m,n}\cos(n\pi) \tag{11}$$

for each m. Two additional constraints involving G_0 and G_1 that result from conservation of volume appear. With truncation (MMAX, NMAX) and utilization of constraints to eliminate the G_m, an eigensystem in $\{A_{m,2n}, E_{m,n}; \lambda_{m,n}\}$ is formed. The contribution from ϕ is due to terms that are even with respect to z. The number of axial modes is (NMAX + 1). The truncated system has [2(NMAX + 1) (MMAX + 1) − 2] equations; (NMAX + 1)(MMAX + 1) originate from the normal force balance, the rest from the kinematic condition. The mth azimuthal mode couples to $(m \pm 1)$ azimuthal modes if $B \neq 0$.

Results

Eigenvalues of the truncated system were found for both even and odd problems, using the subroutine RGG from the EISPAK library (in double precision). Stability of the deformed column ($B \neq 0$) to perturbations is determined by the real part of the eigenvalues; if real $(\lambda_{m,n}) > 0$, then the deformed column is unstable.

As a check on the numerics, the normal mode oscillation frequencies were recovered for the undeformed column. With $B = 0$, azimuthal modes decouple, motion is purely oscillatory, and eigenvalues represent natural frequencies of oscillation. For the odd problem, the algebraic system represents the dispersion relation for modes of the undeformed column that are antisymmetric about $z = 0$.

For the even problem, the situation is more involved since in the $B = 0$ case the additional constraint involving G_1 (present in the $B \neq 0$ case) does not occur. It is necessary to formulate the $B = 0$ case separately. The polynomial term in the ζ_{even} expansion remains necessary for convergence. Azimuthal modes decouple, and oscillation frequencies of the undeformed column are recovered.

Eigenvalues (natural frequencies) of the oscillating undeformed column were recovered to an accuracy of between three and four significant digits, as compared against known values from the closed-form dispersion relation.[4] As system size was increased from $(1152)^2$ to $(1998)^2$, accuracy did not improve. Although Fourier series do not have as rapid convergence characteristics as, say, Chebyshev polynomials,[8] Fourier series do arise naturally in this problem[4,5] and proved to be better expansion functions.

The stability of both the even and odd problems was investigated for a matrix of (B, Λ) values. Values of $B = 0.05$ and 0.005 were used; these correspond to acceleration levels on the order of $(10^{-3})g_{\text{earth}}$ to $(10^{-4})g_{\text{earth}}$ (for nominal values of ρ, γ, and R_0). The Λ values ranged from 1.80 to 3.00. (Smaller Λ values could require a viscous analysis.) Results were checked at increasing truncation values, with no change to stability results. [For the even problem, maximum algebraic system size was $(3118)^2$; for the odd problem, maximum system size was $(1920)^2$.] In each case, real $(\lambda_{m,n}) > 0$ occurred in the eigenvalue results. Thus, external steady lateral forcing was found to be destabilizing for cases considered, i.e., the deformed column was found to be unstable to the time-dependent perturbations; see Tables 1 and 2.

The inherent three-dimensionality with azimuthal mode coupling may preclude the establishment of strictly oscillatory motion at B levels used. Had the deformed column been found stable to the time-dependent perturbations, oscillatory motion could have ensued. For the undeformed ($B = 0$) column, motion is oscillatory, and a balance exists between surface tension/curvature effects and pressure perturbations. This balance is upset for the deformed ($B \neq 0$) column.

An investigation of deformed column stability at smaller B levels taxed the limits of the numerics; further investigation would have required greater computational resources. Also, it is possible that inclusion of viscous effects (which would require a new formulation) could result in stability in some cases.

Table 1 Representative eigenvalues: even

Λ	$B = 0.05$	$B = 0.005$
1.8	(1.22, 7.42)	(0.28, 17.08)
2.0	(1.23, 7.97)	(0.12, 16.82)
2.2	(0.11, 16.59)	(0.86, 16.08)
2.4	(0.24, 14.81)	(0.48, 15.96)
2.6	(0.15, 15.41)	(0.13, 15.85)
2.8	(0.91, 14.73)	(0.90, 13.55)
3.0	(0.77, 17.09)	(0.29, 8.93)

Table 2 Representative eigenvalues: even and odd (B = 0.005)

Λ	Even	Odd
1.8	(0.28, 17.08)	(0.11, 23.46)
2.0	(0.12, 16.82)	(0.03, 26.82)
2.2	(0.86, 16.08)	(0.56, 11.98)
2.4	(0.48, 15.96)	(0.15, 10.82)
2.6	(0.13, 15.85)	(0.04, 21.68)
2.8	(0.90, 13.55)	(0.75, 20.81)
3.0	(0.29, 8.93)	(0.05, 13.51)

Acknowledgments

Part of this work was done while the author was on sabbatical at the National Institute of Standards and Technology (NIST). Discussions wih G. B. McFadden and B. Murray of NIST were most helpful.

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Elastic Response of Accelerating Launch Vehicles Subjected to Varying Control Pulses

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Introduction

MODERN launch vehicles employ large forces for better control performance that result in significant elastic excitation of the lightly damped launch vehicle. These responses cause excitation to payload, corrupt the feedback sensor signal, and produce large dynamic bending moments on the structure. Recently the author considered the effect of trajectory accelerations and found significant changes in the free-vibration behavior of launch vehicles.[1,2] In the present study, the author analyzes the effects of control pulse duration and trajectory accelerations on the overall elastic response of typical launch vehicle geometry.[3] Accelerations at the tip and the maximum dynamic bending moment are obtained for different values of pulse duration in such a way that total momentum of the pulse is constant. The vehicle is modeled as a stepped beam of constant property segments. Transverse vibration of such beams can be adequately modeled using the Euler–Bernoulli beam theory.[2]

Formulation and Solution

The elastic response of a lightly damped structure can be adequately obtained using the mode superposition approach[4] as follows:

$$w(x,t) = \sum_r^{\infty} \frac{1}{\omega_r} \int_0^t F_r(\tau) \exp[-\xi_r \omega_r (t-\tau)] \sin \omega_r (t-\tau)\, d\tau \tag{1}$$

Received Feb. 3, 1996; revision received April 4, 1996; accepted for publication April 25, 1996.

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where ω_r is undamped natural frequency, ξ_r is equivalent viscous damping, and $F_r(\tau)$ is excitation force in rth mode. The modal solution is obtained from the free-vibration equations given by

$$\left(\frac{\partial^4 w_i}{\partial \bar{x}_i^4}\right) - a_i \left(\frac{\partial^2 w_i}{\partial \bar{x}_i^2}\right) + \gamma_i^4 w_i = 0 \tag{2}$$

Here, γ_i $[=(\rho A)_i \omega^2 L_o^4/(EI)_i]^{(1/4)}$ is the dimensionless frequency parameter for the ith segment. Variable $\bar{x}_i$ $(=x_i/L_o)$ takes values from 0 to $\bar{L}_i$ $(=L_i/L_o)$ for all segments, where L_i is the length of each segment and a_i $(=P_i L_o^2/EI_o)$ is the nondimensional axial inertia force parameter due to trajectory acceleration. A new frequency parameter λ for the complete vehicle is defined as

$$\lambda^4 = \left[(\rho A)_o \omega^2 L_o^4 \big/ (EI)_o\right] \tag{3}$$

where ρA_o is the average mass per length and EI_o is the maximum bending rigidity. The general solution[1] of Eq. (2) can be written as

$$w_i = A_i \cosh \lambda_1 \bar{x}_i + B_i \sinh \lambda_1 \bar{x}_i + C_i \cos \lambda_2 \bar{x}_i + D_i \sin \lambda_2 \bar{x}_i \tag{4}$$

where A_i, B_i, C_i, and D_i, are arbitrary constants and λ_1 and λ_2 are roots of characteristic equation for each segment and are obtained as

$$\lambda_1^2 = \frac{\left(a_i^2 + 4\gamma_i^4\right)^{\frac{1}{2}} - a_i}{2} \tag{5}$$

$$\lambda_2^2 = \frac{\left(a_i^2 + 4\gamma_i^4\right)^{\frac{1}{2}} + a_i}{2} \tag{6}$$

Enforcing boundary conditions at two ends and continuity conditions between N segments on displacement, slope, bending moment, and shear force results in a total of $4N$ conditions for $4N$ unknowns and a $4N \times 4N$ characteristic determinant whose zeroes give the natural frequency λ. The corresponding normalized modes are obtained by substituting eigenvalue in the characteristic equations.[1]

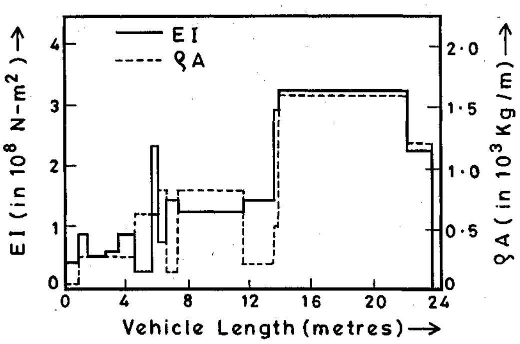

Fig. 1 Structural configuration of a typical launch vehicle.

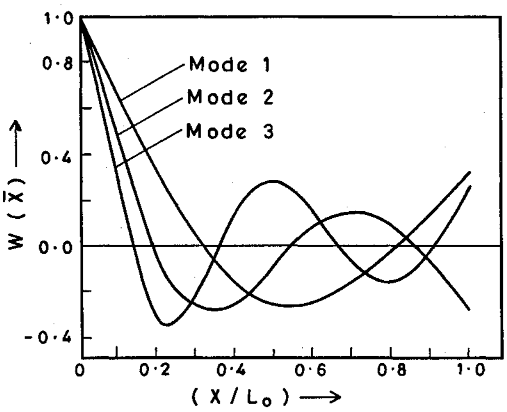

Fig. 2 First three vibration mode shapes of the launch vehicle.